

Universality properties of double series by generalized Walsh system

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Abstract

In this paper we consider a question on existence of double series by generalized Walsh system, which are universal in weighted $L_\mu^1[0, 1]^2$ spaces. In particular, we construct a weighted function $\mu(x, y)$ and a double series by generalized Walsh system of the form

$$\sum_{n,k=1}^{\infty} c_{n,k} \psi_n(x) \psi_k(y) \quad \text{with} \quad \sum_{n,k=1}^{\infty} |c_{n,k}|^q < \infty \text{ for all } q > 2,$$

which is universal in $L_\mu^1[0, 1]^2$ concerning subseries with respect to convergence, in the sense of both spherical and rectangular partial sums.

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1 Introduction

Let X be a Banach space.

Definition 1.1 *A series*

$$\sum_{k=1}^{\infty} f_k, \quad f_k \in X \tag{1}$$

is said to be universal in X with respect to rearrangements, if for any $f \in X$ the members of (1) can be rearranged so that the obtained series $\sum_{k=1}^{\infty} f_{\sigma(k)}$ converges to f by norm of X .

Definition 1.2 *The series (1) is said to be universal (in X) concerning subseries, if for any $f \in X$ it is possible to choose a subseries $\sum_{k=1}^{\infty} f_{n_k}$ from (1), which converges to the f by norm of X .*

Note, that for one-dimensional case there are many papers are devoted to the question on existence of various types of universal series in the sense of convergence almost everywhere and on a measure (see [1] - [9], [11]).

The first usual universal in the sense of convergence almost everywhere trigonometric series were constructed by D.E.Menshov [1] and V.Ya.Kozlov [2]. The series of the form

$$\frac{1}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

was constructed just by them such that for any measurable on $[0, 2\pi]$ function $f(x)$ there exists the growing sequence of natural numbers n_k such that the series having the sequence of partial sums with numbers n_k converges to $f(x)$ almost everywhere on $[0, 2\pi]$.

Note that in this result, when $f(x) \in L^1_{[0, 2\pi]}$, it is impossible to replace convergence almost everywhere by convergence in the metric $L^1_{[0, 2\pi]}$.

This result was distributed by A.A.Talalian on arbitrary orthonormal complete systems . He also established (see [3]), that if $\{\phi_n(x)\}_{n=1}^{\infty}$ - the normalized basis of space $L^p_{[0, 1]}$, $p > 1$, then there exists a series of the form

$$\sum_{k=1}^{\infty} a_k \phi_k(x), \quad a_k \rightarrow 0. \quad (2)$$

which has property: for any measurable function $f(x)$ the members of series (2) can be rearranged so that the again received series converge on a measure on $[0, 1]$ to $f(x)$.

In [4] O. P. Dzangadze these results are transferred to two-dimensional case.

W. Orlicz [5] observed the fact that there exist functional series that are universal with respect to rearrangements in the sense of a.e. convergence in the class of a.e. finite measurable functions. It is also useful to note that even Rieman proved that every convergent numerical series which is not absolutely convergent is universal with respect to rearrangements in the class of all real numbers.

Let $\mu(x)$, $0 < \mu(x) \leq 1$, $x \in [0, 1]$ be a measurable on $[0, 1]$ function and let $L^1_{\mu}[0, 1]$ be a space of real measurable functions $f(x)$, $x \in [0, 1]$ with

$$\int_0^1 |f(x)| \mu(x) dx < \infty.$$

In [6] - [9] it is proved the existence of universal one-dimensional series by trigonometric and classical Walsh system with respect to rearrangements and subseries. Some results for two-dimensional case for classical Walsh system was obtained in [11]. In this paper we consider this problems for double series by generalized Walsh system

2 Preliminary Notes

Let a denote a fixed integer, $a \geq 2$ and put $\omega_a = e^{\frac{2\pi i}{a}}$.

Now we will give the definitions of generalized Rademacher and Walsh systems (see [12]).

Definition 2.1 *The Rademacher system of order a is defined by*

$$\varphi_0(x) = \omega_a^k \text{ if } x \in \left[\frac{k}{a}, \frac{k+1}{a} \right), \quad k = 0, 1, \dots, a-1, \quad x \in [0, 1)$$

and for $n \geq 0$

$$\varphi_n(x+1) = \varphi_n(x) = \varphi_0(a^n x).$$

Definition 2.2 *The generalized Walsh system of order a is defined by*

$$\psi_0(x) = 1,$$

and if $n = \alpha_1 a^{n_1} + \dots + \alpha_s a^{n_s}$ where $n_1 > \dots > n_s$, then

$$\psi_n(x) = \varphi_{n_1}^{\alpha_1}(x) \cdot \dots \cdot \varphi_{n_s}^{\alpha_s}(x).$$

Let's denote the generalized Walsh system of order a by Ψ_a , $a \geq 2$.

Note that Ψ_2 is the classical Walsh system.

The basic properties of the generalized Walsh system of order a are obtained by H.E.Chrestenson, R. Pely, J. Fine, W. Young, C. Vatari, N. Vilenkin and others (see [12]- [17]).

First we present some properties of Ψ_a system (see Definition 2.1).

Property 1. Each n th Rademacher function has period $\frac{1}{a^n}$ and

$$\varphi_n(x) = \text{const} \in \Omega_a = \{1, \omega_a, \omega_a^2, \dots, \omega_a^{a-1}\}, \quad (3)$$

if $x \in \Delta_{n+1}^{(k)} = \left[\frac{k}{a^{n+1}}, \frac{k+1}{a^{n+1}} \right)$, $k = 0, \dots, a^{n+1} - 1$, $n = 1, 2, \dots$.

It is also easily verified, that

$$(\varphi_n(x))^k = (\varphi_n(x))^m, \quad \forall n, k \in \mathcal{N}, \text{ where } m = k \pmod{a} \quad (4)$$

Property 2. It is clear, that for any integer n the Walsh function $\psi_n(x)$ consists of a finite product of Rademacher functions and accepts values from Ω_a .

Property 3. The generalized Walsh system Ψ_a , $a \geq 2$ is a complete orthonormal system in $L^2[0, 1)$ and basis in $L^p[0, 1]$, $p > 1$ (see [5]).

The rectangular and spherical partial sums of the double series

$$\sum_{k, \nu=1}^{\infty} c_{k, \nu} \psi_k(x) \psi_{\nu}(y)$$

will be denoted by

$$S_{n,m}(x, y) = \sum_{k=1}^n \sum_{\nu=1}^m c_{k,\nu} \psi_k(x) \psi_\nu(y)$$

and

$$S_R(x, y) = \sum_{\nu^2 + k^2 \leq R^2} c_{k,\nu} \psi_k(x) \psi_\nu(y).$$

If $g(x, y)$ is a continuous function on $T = [0, 1]^2$, then we set

$$\|g(x, y)\|_C = \max_{(x,y) \in T} |g(x, y)|.$$

3 Main Results

Let's denote the generalized Walsh system of order a by Ψ_a , $a \geq 2$. These are the main results of the paper.

Theorem 3.1 *There exists a double series of the form*

$$\sum_{n,k=1}^{\infty} c_{n,k} \psi_n(x) \psi_k(y) \quad \text{with} \quad \sum_{n,k=1}^{\infty} |c_{n,k}|^q < \infty \quad \text{for all } q > 2 \quad (5)$$

with the following property: for any number $\varepsilon > 0$ a weighted function $\mu(x, y)$ satisfying

$$0 < \mu(x, y) \leq 1, |\{(x, y) \in T : \mu(x, y) \neq 1\}| < \varepsilon \quad (6)$$

can be constructed so that the series (5) is universal in $L_\mu^1(T)$ concerning sub-series with respect to convergence in the sense of both spherical and rectangular partial sums.

Theorem 3.2 *There exists a double series of the form (5) with the following property: for any number $\varepsilon > 0$ a weighted function $\mu(x, y)$ with (6) can be constructed, so that the series (5) is universal in $L_\mu^1(T)$ concerning rearrangements with respect to convergence in the sense of both spherical and rectangular partial sums.*

Repeating the reasoning of the proof of Lemma 2 in [10] we'll receive the following lemma:

Lemma 3.3 *For any given numbers $0 < \varepsilon < 1$, $N_0 > 2$ ($N_0 \in \mathcal{N}$) and a step function*

$$f(x) = \sum_{s=1}^q \gamma_s \cdot \chi_{\Delta_s}(x),$$

where Δ_s is an interval of the form $\Delta_m^{(i)} = \left[\frac{i-1}{2^m}, \frac{i}{2^m} \right]$, $1 \leq i \leq 2^m$, there exist a measurable set $E \subset [0, 1]$ and a polynomial $P(x)$ of the form

$$P(x) = \sum_{k=N_0}^N c_k \psi_k(x)$$

which satisfy the conditions:

$$(1) \quad P(x) = f(x) \quad \text{on } E,$$

$$(2) \quad |E| > (1 - \varepsilon),$$

$$(3) \quad \sum_{k=N_0}^N |c_k|^{2+\varepsilon} < \varepsilon,$$

$$(4) \quad \max_{N_0 \leq m < N} \left[\int_e \left| \sum_{k=N_0}^m c_k \psi_k(x) \right| dx \right] < \varepsilon + \int_e |f(x)| dx,$$

for every measurable subset e of E .

Then applying this Lemma we get next one:

Lemma 3.4 For any numbers $\gamma \neq 0$, $0 < \delta < 1$, $N > 1$ and for any square $\Delta = \Delta_1 \times \Delta_2 \subset T$ there exists a measurable set $E \subset T$ and a polynomial $P(x, y)$ of the form

$$P(x, y) = \sum_{k,s=N}^M c_{k,s} \psi_k(x) \cdot \psi_s(y),$$

with the following properties:

$$(1) \quad |E| > 1 - \delta,$$

$$(2) \quad \sum_{k,s=N}^M |c_{k,s}|^{2+\delta} < \delta,$$

$$(3) \quad P(x, y) = \gamma \cdot \chi_{\Delta}(x, y) \quad \text{for } (x, y) \in E,$$

$$(4) \quad \max_{N \leq \bar{n}, \bar{m} \leq M} \left[\int \int_e \left| \sum_{k,s=N}^{\bar{n}, \bar{m}} c_{k,s} \psi_k(x) \cdot \psi_s(y) \right| dx dy \right] \\ + \max_{\sqrt{2}N \leq R \leq \sqrt{2}M} \left[\int \int_e \left| \sum_{2N^2 \leq k^2 + s^2 \leq R^2} c_{k,s} \psi_k(x) \cdot \psi_s(y) \right| dx dy \right] \leq 16 \cdot |\gamma| \cdot |\Delta|,$$

for every measurable subset e of E .

Proof . We apply Lemma 3.3, setting

$$f(x) = \gamma \cdot \chi_{\Delta_1}(x), \quad N_0 = N, \quad \varepsilon = \frac{\delta}{2}.$$

Then we can define a measurable set $E_1 \subset [0, 1]$ and a polynomial $P_1(x)$ of the form

$$P_1(x) = \sum_{k=N}^{N_1} a_k \psi_k(x)$$

which satisfy the conditions:

$$(1^0) \quad P_1(x) = \gamma \cdot \chi_{\Delta_1}(x) \quad \text{for } x \in E_1,$$

$$(2^0) \quad |E_1| > 1 - \frac{\delta}{2},$$

$$(3^0) \quad \sum_{k=N}^{N_1} |a_k|^{2+\delta} < \delta,$$

$$(4^0) \quad \max_{N \leq \bar{n} \leq N_1} \left[\int_{e_1} \left| \sum_{k=N}^{\bar{n}} a_k \psi_k(x) \right| dx \right] \leq 2 \cdot |\gamma| \cdot |\Delta_1|,$$

for every measurable subset e_1 of E_1 .

Set

$$M_0 = 2 \cdot (N_1^2 + 1) \tag{7}$$

and apply Lemma 3.3 again, setting

$$f(y) = \chi_{\Delta_2}(y), \quad N_0 = M_0, \quad \varepsilon = \frac{\delta}{2}.$$

Then we can define a measurable set $E_2 \subset [0, 1]$ and a polynomial $P_2(y)$ of the form

$$P_2(y) = \sum_{s=M_0}^M b_s \psi_s(y),$$

which satisfy the conditions:

$$(1^{00}) \quad P_2(y) = \chi_{\Delta_2}(y) \quad \text{for } y \in E_2,$$

$$(2^{00}) \quad |E_2| > 1 - \frac{\delta}{2},$$

$$(3^{00}) \quad \sum_{s=M_0}^M |b_s|^{2+\delta} < \delta,$$

$$(4^{00}) \quad \max_{M_0 \leq \bar{m} \leq M} \left[\int_{e_2} \left| \sum_{s=M_0}^{\bar{m}} b_s \psi_s(y) \right| dy \right] \leq 2 \cdot |\Delta_2|,$$

for every measurable subset e_2 of E_2 .

Set

$$E = E_1 \times E_2, \quad (8)$$

$$P(x, y) = P_1(x) \cdot P_2(y) = \sum_{k,s=N}^M c_{k,s} \psi_k(x) \cdot \psi_s(y), \quad (9)$$

where

$$c_{k,s} = a_k \cdot b_s, \quad \text{if } N \leq k \leq N_1, \quad M_0 \leq s \leq M \quad (10)$$

and

$$c_{k,s} = 0, \quad \text{for other } k \text{ and } s.$$

By $(1^0) - (3^0)$, $(1^{00}) - (3^{00})$ and (3.2) - (3.4) we obtain

$$|E| > 1 - \delta,$$

$$\sum_{k,s=N}^M |c_{k,s}|^{2+\delta} = \sum_{k=N}^{N_1} |a_k|^{2+\delta} \cdot \sum_{s=M_0}^M |b_s|^{2+\delta} < \delta,$$

$$P(x, y) = \gamma \cdot \chi_\Delta(x, y) \quad \text{for } (x, y) \in E.$$

Thus, the statements 1) - 3) of Lemma 3.4 are satisfied. Now we will check the fulfillment of statement 4).

Let $N^2 + M_0^2 < R^2 < N_1^2 + M^2$, then for some $m_0 > M_0$ we have $m_0 < R^2 < m_0 + 1$ and from (3.1) it follows, that $R^2 - N_1^2 > (m_0 - 1)^2$.

Consequently taking relations (4^0) , (4^{00}) and (3.2) - (3.4) for any measurable set $e \subset E$ ($e = e_1 \times e_2$, $e_1 \subset E_1$, $e_2 \subset E_2$) we obtain

$$\begin{aligned} & \int \int_e \left| \sum_{N^2 + M^2 \leq k^2 + s^2 \leq R^2} c_{k,s} \psi_k(x) \cdot \psi_s(y) \right| dx dy \\ & \leq \int \int_e \left| \sum_{k=N}^{N_1} \sum_{s=M_0}^{m_0-1} c_{k,s} \psi_k(x) \cdot \psi_s(y) \right| dx dy \\ & + \max_{N < n \leq N_1} \left[\int \int_e \left| \sum_{k=N}^n c_{k,m_0} \psi_k(x) \cdot \psi_{m_0}(y) \right| dx dy \right] \end{aligned}$$

$$\begin{aligned}
&\leq \left[\int_{e_1} \left| \sum_{k=N}^{N_1} a_k \psi_k(x) \right| dx \right] \cdot \left[\int_{e_2} \left| \sum_{s=M_0}^{m_0-1} b_s \psi_s(y) \right| dy \right] \\
&+ |b_{m_0}| \cdot \left[\int_{e_2} |\psi_{m_0}(y)| dy \right] \cdot \max_{N < n \leq N_1} \left[\int_{e_1} \left| \sum_{k=N}^n a_k \psi_k(x) \right| dx \right] \\
&\leq 12 \cdot |\gamma| \cdot |\Delta|.
\end{aligned}$$

Similarly, for $N \leq \bar{n} \leq N_1$, $M_0 \leq \bar{m} \leq M$, we get

$$\int \int_e \left| \sum_{k,s=N}^{\bar{n}, \bar{m}} c_{k,s} \psi_k(x) \cdot \psi_s(y) \right| dx dy \leq 4 \cdot |\gamma| \cdot |\Delta|.$$

Lemma 3.4 is proved.

Lemma 3.5 For any numbers $\varepsilon > 0$, $N > 1$ and a step function

$$f(x, y) = \sum_{\nu=1}^{\nu_0} \gamma_\nu \cdot \chi_{\Delta_\nu}(x, y),$$

there exists a measurable set $E \subset T$ and a polynomial $P(x, y)$ of the form

$$P(x, y) = \sum_{k,s=N}^M c_{k,s} \psi_k(x) \cdot \psi_s(y),$$

which satisfy the following conditions:

$$(1^0) \quad P(x, y) = f(x, y) \quad \text{for } (x, y) \in E,$$

$$(2^0) \quad |E| > 1 - \varepsilon,$$

$$(3^0) \quad \sum_{k,s=N}^M |c_{k,s}|^{2+\varepsilon} < \varepsilon,$$

$$\begin{aligned}
(4^0) \quad &\max_{N \leq \bar{n}, \bar{m} < M} \left[\int \int_e \left| \sum_{k,s=N}^{\bar{n}, \bar{m}} c_{k,s} \psi_k(x) \cdot \psi_s(y) \right| dx dy \right] \\
&+ \max_{\sqrt{2}N \leq R \leq \sqrt{2}M} \left[\int \int_e \left| \sum_{2N^2 \leq k^2 + s^2 \leq R^2} c_{k,s} \psi_k(x) \cdot \psi_s(y) \right| dx dy \right] \\
&\leq 2 \cdot \int \int_e |f(x, y)| dx dy + \varepsilon,
\end{aligned}$$

for every measurable subset e of E .

Proof . Without any loss of generality, we assume that

$$\max_{1 \leq \nu \leq \nu_0} (|\gamma_\nu| \cdot |\Delta_\nu|) < \frac{\varepsilon}{32}, \quad (11)$$

(Δ_ν , $1 \leq \nu \leq \nu_0$ are the constancy rectangular domian of $f(x, y)$, i.e. where the function $f(x, y)$ is constant).

Given an integer $1 \leq \nu \leq \nu_0$, by applying Lemma 3.4 with $\delta = \frac{\varepsilon}{16\nu_0}$, we find that there exists a measurable set $E_\nu \subset T$ and a polynomial $P_\nu(x, y)$ of the form

$$P_\nu(x, y) = \sum_{k,s=N_\nu}^{M_\nu} c_{k,s}^{(\nu)} \psi_k(x) \cdot \psi_s(y) \quad (12)$$

with the following properties:

$$|E_\nu| > 1 - \frac{\varepsilon}{2^\nu}, \quad (13)$$

$$\sum_{k,s=N_\nu}^{M_\nu} |c_{k,s}^{(\nu)}|^{2+\varepsilon} < \frac{\varepsilon}{\nu_0}, \quad (14)$$

$$P_\nu(x, y) = \gamma_\nu \cdot \chi_{\Delta_\nu}(x, y) \quad \text{for } (x, y) \in E_\nu, \quad (15)$$

$$\begin{aligned} & \max_{N_\nu \leq \bar{n}, \bar{m} \leq M_\nu} \left[\int \int_e \left| \sum_{k,s=N_\nu}^{\bar{n}, \bar{m}} c_{k,s}^{(\nu)} \psi_k(x) \cdot \psi_s(y) \right| dx dy \right] \\ & + \max_{\sqrt{2}N_\nu \leq R \leq \sqrt{2}M_\nu} \left[\int \int_e \left| \sum_{2N_\nu^2 \leq k^2 + s^2 \leq R^2} c_{k,s}^{(\nu)} \psi_k(x) \cdot \psi_s(y) \right| dx dy \right] \\ & \leq 16 \cdot |\gamma_\nu| \cdot |\Delta_\nu| < \frac{\varepsilon}{2}, \end{aligned} \quad (16)$$

for every measurable subset e of E_ν (see (11)).

Then we can take

$$N_1 = N, \quad n_\nu = M_{\nu-1} + 1, \quad 1 \leq \nu \leq \nu_0.$$

Set

$$E = \bigcap_{\nu=1}^{\nu_0} E_\nu, \quad (17)$$

$$P(x, y) = \sum_{\nu=1}^{\nu_0} P_\nu(x, y) = \sum_{k,s=N}^M c_{k,s} \psi_k(x) \cdot \psi_s(y), \quad M = M_{\nu_0}, \quad (18)$$

where

$$c_{k,s} = c_{k,s}^{(\nu)}, \quad \text{for } N_\nu \leq k, s \leq M_\nu, \quad 1 \leq \nu \leq \nu_0 \quad (19)$$

and

$$c_{k,s} = 0, \quad \text{for other } k \text{ and } s.$$

From (13) - (15), (17) - (19) we obtain:

$$P(x, y) = f(x, y) \text{ for } (x, y) \in E,$$

$$|E| > 1 - \varepsilon,$$

$$\sum_{k,s=N}^M |c_{k,s}|^{2+\varepsilon} < \sum_{\nu=1}^{\nu_0} \left[\sum_{k,s=N_\nu}^{M_\nu} |c_{k,s}^{(\nu)}|^{2+\varepsilon} \right] < \varepsilon$$

Then, let $R \in [\sqrt{2}N, \sqrt{2}M]$, then for some $\nu', 1 \leq \nu' \leq \nu_0$ we have $\sqrt{2}N_{\nu'} \leq R \leq \sqrt{2}N_{\nu'+1}$, consequently from (18) and (19) we have

$$\sum_{2N^2 \leq k^2+s^2 \leq R^2} c_{k,s} \psi_k(x) \cdot \psi_s(y) = \sum_{\nu=1}^{\nu'-1} P_\nu(x, y)$$

$$+ \sum_{2N_{\nu'}^2 \leq k^2+s^2 \leq R^2} c_{k,s}^{(\nu')} \psi_k(x) \cdot \psi_s(y).$$

In view of the conditions (13) - (16) and the equality $P(x, y) = f(x, y)$ on E , for any measurable set $e \subset E$ we obtain

$$\int \int_e \left| \sum_{2N^2 \leq k^2+s^2 \leq R^2} c_{k,s} \psi_k(x) \cdot \psi_s(y) \right| dx dy$$

$$\leq \int \int_e \left| \sum_{\nu=1}^{\nu'-1} P_\nu(x, y) \right| dx dy$$

$$+ \int \int_e \left| \sum_{2N_{\nu'}^2 \leq k^2+s^2 \leq R^2} c_{k,s}^{(\nu')} \psi_k(x) \cdot \psi_s(y) \right| dx dy$$

$$\leq \int \int_e |f(x, y)| dx dy + \frac{\varepsilon}{2}.$$

Similarly, for any $e \subset E$ we have

$$\max_{N \leq \bar{n}, \bar{m} \leq M} \left[\int \int_e \left| \sum_{k,s=N}^{\bar{n}, \bar{m}} c_{k,s} \psi_k(x) \cdot \psi_s(y) \right| dx dy \right]$$

$$\leq \int \int_e |f(x, y)| dx dy + \frac{\varepsilon}{2}.$$

Lemma 3.5 is proved.

4 Proofs of the theorems

The Theorem 3.1 is proved similarly Theorem 3 in [11], but for maintenance of integrity of this paper, here we will give the proof :

Proof of Theorem 3.1.

Let

$$\{f_s(x, y)\}_{s=1}^{\infty}, \quad (x, y) \in T \quad (20)$$

be a sequence of all step functions, values and constancy interval endpoints of which are rational numbers. Applying Lemma 3.5 consecutively, we can find a sequence $\{E_s\}_{s=1}^{\infty}$ of sets and a sequence of polynomials

$$P_s(x, y) = \sum_{k, \nu=N_{s-1}}^{N_s-1} c_{k, \nu}^{(s)} \psi_k(x) \psi_{\nu}(y), \quad (21)$$

$$1 = N_0 < N_1 < \dots < N_s < \dots, \quad s = 1, 2, \dots,$$

which satisfy the conditions:

$$P_s(x, y) = f_s(x, y), \quad (x, y) \in E_s, \quad (22)$$

$$|E_s| > 1 - 2^{-2(s+1)}, \quad E_s \subset T, \quad (23)$$

$$\sum_{k, \nu=N_{s-1}}^{N_s-1} |c_{k, \nu}^{(s)}|^{2+2^{-2s}} < 2^{-2s}, \quad (24)$$

$$\begin{aligned} & \max_{N_{s-1} \leq \bar{n}, \bar{m} < N_s} \left[\int \int_e \left| \sum_{k, \nu=N_{s-1}}^{\bar{n}, \bar{m}} c_{k, \nu}^{(s)} \psi_k(x) \cdot \psi_{\nu}(y) \right| dx dy \right] \\ & + \max_{\sqrt{2}N_{s-1} \leq R \leq \sqrt{2}N_s} \left[\int \int_e \left| \sum_{2N_{s-1}^2 \leq k^2 + \nu^2 \leq R^2} c_{k, \nu}^{(s)} \psi_k(x) \cdot \psi_{\nu}(y) \right| dx dy \right] \\ & \leq 2 \cdot \int \int_e |f_s(x, y)| dx dy + 2^{-2(s+1)}, \end{aligned} \quad (25)$$

for every measurable subset e of E_s .

Denote

$$\sum_{k, \nu=1}^{\infty} c_{k, \nu} \psi_k(x) \psi_{\nu}(y) = \sum_{s=1}^{\infty} \left[\sum_{k, \nu=N_{s-1}}^{N_s-1} c_{k, \nu}^{(s)} \psi_k(x) \psi_{\nu}(y) \right], \quad (26)$$

where

$$c_{k, \nu} = c_{k, \nu}^{(s)}, \quad \text{for } N_{s-1} \leq k, \nu < N_s, \quad s = 1, 2, \dots$$

For an arbitrary number $\varepsilon > 0$ we set

$$\begin{cases} \Omega_n = \bigcap_{s=n}^{\infty} E_s, & n = 1, 2, \dots; \\ E = \Omega_{n_0} = \bigcap_{s=n_0}^{\infty} E_s, & n_0 = [\log_{1/2} \varepsilon] + 1; \\ B = \bigcup_{n=n_0}^{\infty} \Omega_n = \Omega_{n_0} \cup \left(\bigcup_{n=n_0+1}^{\infty} \Omega_n \setminus \Omega_{n-1} \right). \end{cases} \quad (27)$$

It is obvious (see (23), (27)) that $|B| = 1$ and $|E| > 1 - \varepsilon$.

We define a function $\mu(x, y)$ in the following way:

$$\mu(x, y) = \begin{cases} 1, & \text{for } (x, y) \in E \cup (T \setminus B); \\ \mu_n, & \text{for } (x, y) \in \Omega_n \setminus \Omega_{n-1}, \quad n \geq n_0 + 1, \end{cases} \quad (28)$$

where

$$\begin{aligned} \mu_n &= \left[2^{2n} \cdot \prod_{s=1}^n h_s \right]^{-1}; \\ h_s &= \|f_s\|_C + \max_{N_{s-1} \leq \bar{n}, \bar{m} < N_s} \left\| \sum_{k, \nu=N_{s-1}}^{\bar{n}, \bar{m}} c_{k, \nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right\|_C \\ &+ \max_{\sqrt{2}N_{s-1} \leq R \leq \sqrt{2}N_s} \left\| \sum_{2N_{s-1}^2 \leq k^2 + \nu^2 \leq R^2} c_{k, \nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right\|_C + 1. \end{aligned} \quad (29)$$

From (24), (26) - (29) we obtain

(A) - $0 < \mu(x, y) \leq 1$, $\mu(x, y)$ is a measurable function and

$$|\{(x, y) \in T : \mu(x, y) \neq 1\}| < \varepsilon.$$

(B) - $\sum_{k, \nu=1}^{\infty} |c_{k, \nu}|^q < \infty$ for all $q > 2$.

Hence, obviously we have (see (24) and (26))

$$\lim_{\min\{k, \nu\} \rightarrow \infty} c_{k, \nu} = 0.$$

It follows from (27) - (29) that for all $s \geq n_0$ and $N_{s-1} \leq \bar{n}, \bar{m} < N_s$

$$\begin{aligned} & \int \int_{T \setminus \Omega_s} \left| \sum_{k, \nu=N_{s-1}}^{\bar{n}, \bar{m}} c_{k, \nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right| \mu(x, y) dx dy \\ &= \sum_{n=s+1}^{\infty} \left[\int \int_{\Omega_n \setminus \Omega_{n-1}} \left| \sum_{k, \nu=N_{s-1}}^{\bar{n}, \bar{m}} c_{k, \nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right| \mu_n dx dy \right] \end{aligned}$$

$$\leq \sum_{n=s+1}^{\infty} 2^{-2n} \left[\int \int_T \left| \sum_{k,\nu=N_{s-1}}^{\bar{n},\bar{m}} c_{k,\nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right| h_s^{-1} dx dy \right] < \frac{1}{3} 2^{-2s}. \quad (30)$$

Analogously for all $s \geq n_0$ and $\sqrt{2}N_{s-1} \leq R \leq \sqrt{2}N_s$ we have

$$\int \int_{T \setminus \Omega_s} \left| \sum_{2N_{s-1}^2 \leq k^2 + \nu^2 \leq R^2} c_{k,\nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right| \mu(x, y) dx dy < \frac{1}{3} 2^{-2s}. \quad (31)$$

By (21), (27) - (29) for all $s \geq n_0$ we have

$$\begin{aligned} & \int \int_T |P_s(x, y) - f_s(x, y)| \mu(x, y) dx dy \\ &= \int \int_{\Omega_s} |P_s(x, y) - f_s(x, y)| \mu(x, y) dx dy \\ &+ \int \int_{T \setminus \Omega_s} |P_s(x, y) - f_s(x, y)| \mu(x, y) dx dy \\ &= \sum_{n=s+1}^{\infty} \left[\int \int_{\Omega_n \setminus \Omega_{n-1}} |P_s(x, y) - f_s(x, y)| \mu_n dx dy \right] \\ &\leq \sum_{n=s+1}^{\infty} 2^{-2n} \left[\int \int_T \left(|f_s(x, y)| + \sum_{k,\nu=N_{s-1}}^{N_s-1} c_{k,\nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right) h_s^{-1} dx dy \right] \\ &< \frac{1}{3} 2^{-2s} < 2^{-2s}. \end{aligned} \quad (32)$$

By (25) and (27) - (30) for all $N_{s-1} \leq \bar{n}, \bar{m} < N_s$ and $s \geq n_0 + 1$ we obtain

$$\begin{aligned} & \int \int_T \left| \sum_{k,\nu=N_{s-1}}^{\bar{n},\bar{m}} c_{k,\nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right| \mu(x, y) dx dy \\ & \int \int_{\Omega_s} \left| \sum_{k,\nu=N_{s-1}}^{\bar{n},\bar{m}} c_{k,\nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right| \mu(x, y) dx dy \\ & \int \int_{T \setminus \Omega_s} \left| \sum_{k,\nu=N_{s-1}}^{\bar{n},\bar{m}} c_{k,\nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right| \mu(x, y) dx dy \\ &< \sum_{n=n_0+1}^s \left[\int \int_{\Omega_n \setminus \Omega_{n-1}} \left| \sum_{k,\nu=N_{s-1}}^{\bar{n},\bar{m}} c_{k,\nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right| \cdot \mu_n dx dy \right] + \frac{1}{3} 2^{-2s} \\ &< \sum_{n=n_0+1}^s \left(2^{-2(s+1)} + 2 \cdot \int \int_{\Omega_n \setminus \Omega_{n-1}} |f_s(x, y)| dx dy \right) \cdot \mu_n + \frac{1}{3} 2^{-2s} \end{aligned}$$

$$\begin{aligned}
&= 2^{-2(s+1)} \cdot \sum_{n=n_0+1}^s \mu_n + \int \int_{\Omega_s} |f_s(x, y)| \mu(x, y) dx dy + \frac{1}{3} 2^{-2s} \\
&< 2 \cdot \int \int_T |f_s(x, y)| \mu(x, y) dx dy + 2^{-2s}.
\end{aligned} \tag{33}$$

Analogously for all $s \geq n_0$ and $\sqrt{2}N_{s-1} \leq R \leq \sqrt{2}N_s$ we have (see (31))

$$\begin{aligned}
&\int \int_T \left| \sum_{2N_{s-1}^2 \leq k^2 + \nu^2 \leq R^2} c_{k,\nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right| \mu(x, y) dx dy \\
&< 2 \cdot \int \int_T |f_s(x, y)| \mu(x, y) dx dy + 2^{-2s}.
\end{aligned} \tag{34}$$

Now we'll show that the series (26) is universal in $L_\mu^1(T)$ concerning subseries with respect to convergence by both spherical and rectangular partial sums.

Let $f(x, y) \in L_\mu^1(T)$, i. e.

$$\int \int_T |f(x, y)| \mu(x, y) dx dy < \infty.$$

It is easy to see that we can choose a function $f_{n_1}(x, y)$ from the sequence (20) such that

$$\int \int_T |f(x, y) - f_{n_1}(x, y)| \mu(x, y) dx dy < 2^{-2}, \quad n_1 > n_0 + 1. \tag{35}$$

Hence, we have

$$\int \int_T |f_{n_1}(x, y)| \mu(x, y) dx dy < 2^{-2} + \int \int_T |f(x, y)| \mu(x, y) dx dy. \tag{36}$$

From (33) and (35) we get

$$\begin{aligned}
&\int \int_T |f(x, y) - P_{n_1}(x, y)| \mu(x, y) dx dy \\
&\leq \int \int_T |f(x, y) - f_{n_1}(x, y)| \mu(x, y) dx dy \\
&+ \int \int_T |f_{n_1}(x, y) - P_{n_1}(x, y)| \mu(x, y) dx dy < 2 \cdot 2^{-2}.
\end{aligned}$$

Assume that numbers $n_1 < n_2 < \dots < n_{q-1}$ are chosen in such a way that the following condition is satisfied:

$$\int \int_T \left| f(x, y) - \sum_{s=1}^j P_{n_s}(x, y) \right| \mu(x, y) dx dy < 2 \cdot 2^{-2j}, \quad 1 \leq j \leq q-1. \tag{37}$$

Now we choose a function $f_{n_q}(x, y)$ from the sequence (20) such that

$$\begin{aligned} \int \int_T \left| \left(f(x, y) - \sum_{s=1}^{q-1} P_{n_s}(x, y) \right) - f_{n_q}(x, y) \right| \mu(x, y) dx dy \\ < 2 \cdot 2^{-2q}, \quad n_q > n_{q-1}. \end{aligned} \quad (38)$$

This with (37) imply

$$\int \int_T |f_{n_q}(x, y)| \mu(x, y) dx dy < 2^{-2q} + 2 \cdot 2^{-2(q-1)} = 9 \cdot 2^{-2q}. \quad (39)$$

Hence and from (21), (32) - (34) we obtain

$$\int \int_T |f_{n_q}(x, y) - P_{n_q}(x, y)| \mu(x, y) dx dy < 2^{-2n_q}, \quad (40)$$

where

$$\begin{aligned} P_{n_q}(x, y) &= \sum_{k, \nu=N_{n_q-1}}^{N_{n_q}-1} c_{k, \nu}^{(n_q)} \psi_k(x) \psi_\nu(y), \\ \max_{N_{n_q-1} \leq \bar{n}, \bar{m} < N_{n_q}} \left[\int \int_T \left| \sum_{k, \nu=N_{n_q-1}}^{\bar{n}, \bar{m}} c_{k, \nu}^{(n_q)} \psi_k(x) \cdot \psi_\nu(y) \right| \mu(x, y) dx dy \right] \\ &< 19 \cdot 2^{-2q}. \end{aligned} \quad (41)$$

Analogously we have

$$\begin{aligned} \max_{\sqrt{2}N_{n_q-1} \leq R \leq \sqrt{2}N_{n_q}} \left[\int \int_T \left| \sum_{2N_{n_q-1}^2 \leq k^2 + \nu^2 \leq R^2} c_{k, \nu}^{(n_q)} \psi_k(x) \cdot \psi_\nu(y) \right| \mu(x, y) dx dy \right] \\ < 19 \cdot 2^{-2q}. \end{aligned}$$

In quality subseries of Theorem we shall take

$$\sum_{q=1}^{\infty} P_{n_q}(x, y) = \sum_{q=1}^{\infty} \left[\sum_{k, \nu=N_{n_q-1}}^{N_{n_q}-1} c_{k, \nu}^{(n_q)} \psi_k(x) \psi_\nu(y) \right]. \quad (42)$$

From (38) and (39) we have

$$\begin{aligned} \int \int_T \left| f(x, y) - \sum_{s=1}^q P_{n_s}(x, y) \right| \mu(x, y) dx dy \\ \leq \int \int_T \left| \left(f(x, y) - \sum_{s=1}^{q-1} P_{n_s}(x, y) \right) - f_{n_q}(x, y) \right| \mu(x, y) dx dy \end{aligned}$$

$$+ \int \int_T |f_{n_q}(x, y) - P_{n_q}(x, y)| \mu(x, y) dx dy < 2 \cdot 2^{-2q}. \quad (43)$$

Let \bar{n} and \bar{m} be arbitrary natural numbers. Then for some natural number q we have

$$N_{n_q-1} \leq \min\{\bar{n}, \bar{m}\} < N_{n_q}.$$

Taking into account (40) and (42) for rectangular partial sums $S_{\bar{n}, \bar{m}}(x, y)$ of (41) we get

$$\begin{aligned} & \int \int_T |S_{\bar{n}, \bar{m}}(x, y) - f(x, y)| \mu(x, y) dx dy \\ & \leq \int \int_T \left| f(x, y) - \sum_{s=1}^q P_{n_s}(x, y) \right| \mu(x, y) dx dy \\ & + \max_{N_{n_q-1} \leq \bar{n}, \bar{m} < N_{n_q}} \left[\int \int_T \left| \sum_{k, \nu=N_{n_q-1}}^{\bar{n}, \bar{m}} c_{k, \nu}^{(n_q)} \psi_k(x) \cdot \psi_\nu(y) \right| \mu(x, y) dx dy \right] \\ & < 21 \cdot 2^{-2q}. \end{aligned} \quad (44)$$

Analogously for $\sqrt{2}N_{n_q-1} \leq R \leq \sqrt{2}N_{n_q}$ we have

$$\int \int_T |S_R(x, y) - f(x, y)| \mu(x, y) dx dy < 21 \cdot 2^{-2q}, \quad (45)$$

where $S_R(x, y)$ the spherical partial sums of (41).

From (44) and (45) we conclude that the series (26) is universal in $L_\mu^1(T)$ concerning subseries with respect to convergence by both spherical and rectangular partial sums (see Definition 1.2).

Theorem 3.1 is proved.

Remark. We can show Theorem 3.2 by the method in the proof of Theorem 3.1.

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